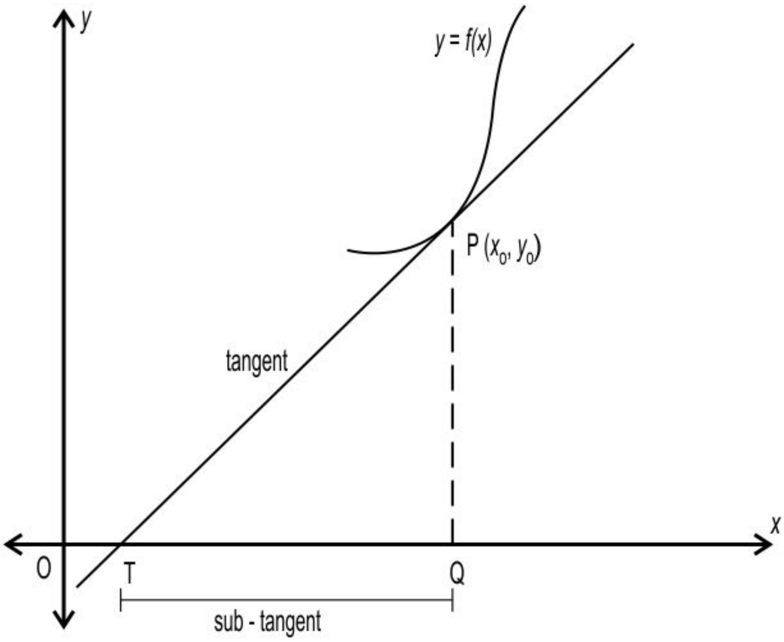
Application of Derivatives and multiconcepts

Multiple Choice Questions

Q: 1 **The sub-tangent of a curve at a point is the projection on the *x* -axis of the portion of the tangent to the curve between the *x* -axis and the point of tangency. The sub-tangent of a curve *y* = *f* (*x*) at a point P(x_o , y_o) is illustrated below.**



Among the given slopes of tangents of a curve at a given point, which will result in the longest sub-tangent?

1 30°

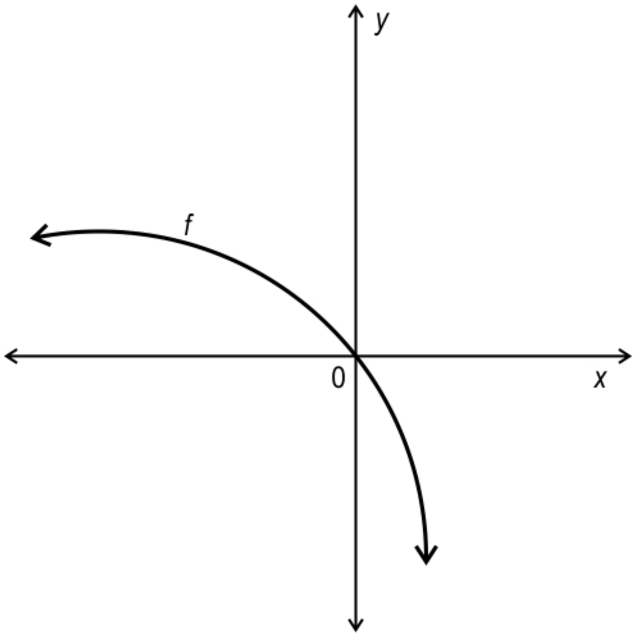
2 45°

3 60°

4 90°

Free Response Questions

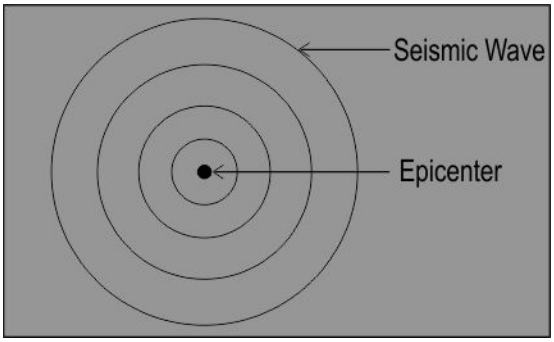
Q: 2 **The graph of a function, *f* : R -> R, is shown below.** **[1]**



Nadeem said that *f* '(0) > *f* (0).

Is Nadeem right? Give a valid reason.

Q: 3 **During an earthquake, seismic waves radiate from the epicenter of an earthquake in a circular pattern, as shown in the figure below.** **[1]**



If seismic waves travel at a speed of approximately 6 km/sec, then what is the rate of change of the area affected by the earthquake when the radius of the affected area is 25 km? Show your steps.

(Note: Take π as 3.14.)

Q: 4 f(x) is an increasing function on the interval [0, 4], and f'(5) > 0. **[1]**

Based on this information, is f(x) an increasing function on the interval [0, 5]? Justify your answer.

Q: 5 A drone is an unmanned remotely operated aerial vehicle, often used for target practice or surveillance. **[2]**

One such drone is flying according to the equation $s = t^n + 10$, where s is the distance of the drone from its remote's location at time t and n is a real number. The position of the remote is fixed.

If the velocity of the drone is equal to its acceleration at 3 seconds, find n. Show your work.

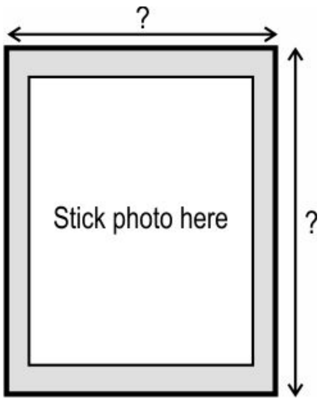
Q: 6 Find the equation of the tangent to the curve $x^2 y^2 = 4$ at the point (1, -2). Show your steps. **[2]**

Q: 7 When the diameter of a circle is 8 cm, by what factor does a small change in diameter affect its area? Show your work. **[2]**

Q: 8 Mr Aithal, a mathematics teacher, announces the following activity in his classroom and assures grand prizes for the winners. **[5]**

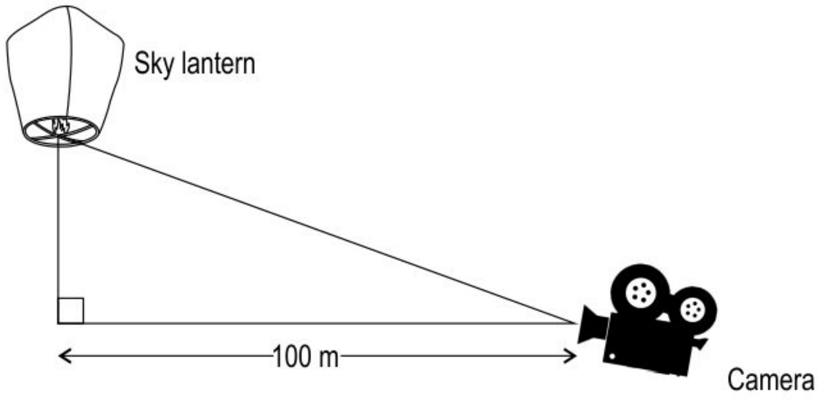
Instructions:

- ♦ Make a rectangular photo frame of total area 80 cm^2 using a chart paper.
- ♦ The frame should have a margin of 1.25 cm each at the top and the bottom.
- ♦ The frame should have a margin of 1 cm each on the left and the right sides.
- ♦ The area available at the centre to stick the photo should be maximum.



What must be the dimensions of such a photo frame? Show your work.

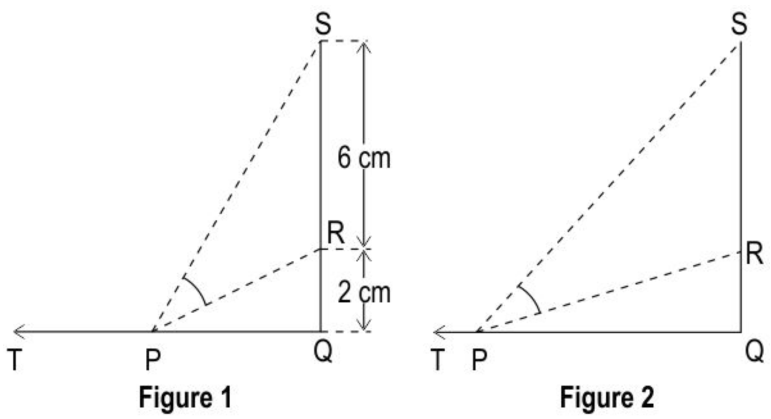
Q: 9 **A camera positioned on the ground recorded a sky lantern that was located 100 meters away. The lantern raised vertically from the ground into the sky at a constant rate of 25 meters per minute, and this entire process was captured on camera.** **[5]**



(Note: The figure is not to scale.)

When the lantern was at 75 m from the ground, what was the rate of change of angle of elevation, in radians/min? Show your steps.

Q: 10 **In the figures shown below, points Q, R and S are fixed. Point P can move forward and backward along ray QT.** **[5]**



(Note: The figures are not to scale.)

What should be the length of PQ such that ∠RPS is maximum? Show your work.

Case Study

Answer the questions based on the given information.

The total cost $C(n)$ of manufacturing n earphone sets per day in the House of Spark Electronics Limited is given by:

$C(n) = 400 + 4n + 0.0001n^2$ dollars.

Each earphone set is sold at:

$q = 10 - 0.0004\, n$ dollars where $n \geq 0$, $q \geq 0$

The daily profit in dollars is determined by the equation:

$P(n) = qn - C(n)$

Q: 11 **The marginal cost, $M(n)$ is the change in total production cost that comes from making or producing one additional unit. It is determined by the instantaneous rate of change of the total cost.** **[2]**

Find the marginal cost $M(n)$ of 10 earphone sets. Show your work.

Q: 12 **What quantity of daily production maximizes the profit? Show your work.** **[3]**

The table below gives the correct answer for each multiple-choice question in this test.

Q.No	Correct Answers
1	1

Q.No	What to look for	Marks
2	Writes that Nadeem is wrong.	0.5
	Gives a reason. For example, $f(0) = 0$, and since f is decreasing, $f'(0) < 0$. Hence, $f'(0) < f(0)$.	0.5
3	Finds the rate of change of the affected area as $\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$ km^2/sec.	0.5
	Finds the rate of change of the area affected by the earthquake when the radius of the affected area (r) is 25 km as $2 \times 3.14 \times 25 \times 6 = 942$ km^2/sec.	0.5
4	Writes that f(x) need not be an increasing function on [0, 5].	0.5
	Gives a reason. For example, even though $f'(5) > 0$, there may be an x in (4, 5), such that $f'(x) < 0$.	0.5
5	Differentiates s with respect to time to find the velocity of the drone as: $s' = nt^{(n-1)}$	0.5
	Differentiates s' with respect to time to find the acceleration of the drone as: $s'' = n(n-1)t^{(n-2)}$	0.5
	Equates velocity of the drone to its acceleration at 3 seconds to find n as: $n \times 3^{(n-1)} = n(n-1) \times 3^{(n-2)}$ $\Rightarrow n \times 3^{(n-1)} = n(n-1) \times 3^{(n-1)} \times 3^{-1}$ $\Rightarrow n = 4$	1
6	Differentiates the given equation using the chain rule as follows: $2xy^2 + (x^2)\,2y\left(\frac{dy}{dx}\right) = 0$	0.5
	Simplifies the above equation as: $\frac{dy}{dx} = \frac{-y}{x}$	0.5

Q.No	What to look for	Marks
	Substitutes (1, -2) in the above equation to get the value of slope (m) as 2.	0.5
	Finds the equation of tangent as: $y - (-2) = 2(x - 1)$ $\Rightarrow y = 2x - 4$	0.5
7	Writes the area of a circle in terms of diameter, D as: $A = \frac{\pi}{4} D^2$	0.5
	Finds the rate of change of area with respect to diameter as follows: $\frac{dA}{dD} = \frac{\pi}{2} D$	1
	Finds $\frac{dA}{dD}$ when D = 8 cm as 4π cm^2/cm. Concludes that for a small change in diameter, the area changes by a factor of 4π. (Award full marks if the problem is solved correctly using approximation concept to obtain 4π x as the answer, where x is the small change in diameter.)	0.5
8	Assumes the width of photo frame to be x cm and its length to be $\frac{80}{x}$ cm.	0.5
	Subtracts the specified margins from the width and length to find the area (A) available to stick the photo as: $A = (x - 2)(\frac{80}{x} - \frac{5}{2})$ $\Rightarrow A = 85 - \frac{5}{2}x - \frac{160}{x}$	1
	Differentiates area with respect to x as: $\frac{dA}{dx} = -\frac{5}{2} + \frac{160}{x^2}$	1

Q.No	What to look for	Marks
	Equates the above derivative to zero and finds the critical point as: $-\frac{5}{2} + \frac{160}{x^2} = 0$ $\Rightarrow x^2 = 64$ $\Rightarrow x = 8$ (as x being a length cannot be negative)	**1**
	Finds $\frac{d^2A}{dx^2}$ at $x = 8$ as: $\frac{d^2A}{dx^2}_{(\text{at } x=8)} = -\frac{320}{x^3}_{(\text{at } x=8)} = -\frac{5}{8} < 0$ **Concludes that by second derivative test, the area is maximum at $x = 8$ cm.**	**1**
	Finds the length of the frame as $\frac{80}{8} = 10$ cm. **Concludes that the required dimensions of the photo frame are 8 cm and 10 cm.**	**0.5**
9	**Takes x as the distance between the lantern and the ground, θ as camera's angle of elevation in radians and t as the time in minutes.** **Writes that $\frac{dx}{dt} = 25$ m/min.**	**0.5**
	Uses tangent function and writes: $\tan\theta = \frac{x}{100}$	**0.5**
	Differentiates the above equation with respect to t to get: $\sec^2\theta \times \frac{d\theta}{dt} = \frac{1}{100} \times \frac{dx}{dt}$	**1**
	Uses steps 1 and 3 to write: $\frac{d\theta}{dt} = \frac{1}{4\sec^2\theta}$	**0.5**

Q.No	What to look for	Marks
	Uses secant function and writes: **sec θ =** $\frac{y}{100}$ **where *y* is the distance between the camera and the lantern.**	**0.5**
	Uses the Pythagoras theorem to find *y* as: $y^2 = x^2 + 100^2$ $\Rightarrow y^2 = 75^2 + 100^2$ $\Rightarrow y = 125$, as $y > 0$.	**1**
	Substitutes *y* = 125 to get sec θ as $\frac{5}{4}$ **.**	**0.5**
	Substitutes the value of sec θ in the equation obtained in step 3 to get: $\frac{d\theta}{dt} = \frac{4}{25}$ **or 0.16 radians/min.**	**0.5**
10	**Considers the length of PQ as *x* cm and finds ∠QPR and ∠QPS as:** $\angle QPR = \tan^{-1}\left(\frac{2}{x}\right)$ $\angle QPS = \tan^{-1}\left(\frac{8}{x}\right)$	**0.5**
	Considers ∠RPS as θ finds θ in terms of *x* as: $\theta = \tan^{-1}\left(\frac{8}{x}\right) - \tan^{-1}\left(\frac{2}{x}\right)$	**0.5**
	Finds the derivative of θ with respect to *x* as follows: $\frac{d\theta}{dx} = \frac{d}{dx}\left(\tan^{-1}\left(\frac{8}{x}\right) - \tan^{-1}\left(\frac{2}{x}\right)\right)$ $= \frac{d}{dx}\left(\tan^{-1}\left(\frac{8}{x}\right)\right) - \frac{d}{dx}\left(\tan^{-1}\left(\frac{2}{x}\right)\right)$	**0.5**

Q.No	What to look for	Marks
	Applies chain rule and differentiates as follows: $\frac{1}{\left(\frac{8}{x}\right)^2+1} \cdot \frac{d}{dx}\left(\frac{8}{x}\right) - \frac{1}{\left(\frac{2}{x}\right)^2+1} \cdot \frac{d}{dx}\left(\frac{2}{x}\right)$ $= \frac{2}{\left[\left(\frac{4}{x^2}+1\right)x^2\right]} - \frac{8}{\left[\left(\frac{64}{x^2}+1\right)x^2\right]}$	**1.5**
	Equates the derivative to 0 to find that the maximum value of θ will occur when *x* is either 4 or (-4). States that the minimum value of θ is 0, which cannot occur at *x* = 4 and hence it must be the maxima. The working may look as follows: $\frac{8}{\left[\left(\frac{64}{x^2}+1\right)x^2\right]} = \frac{2}{\left[\left(\frac{4}{x^2}+1\right)x^2\right]}$ Cancelling x^2 on both sides, $4\left(\frac{4}{x^2} + 1\right) = \frac{64}{x^2} + 1$ Rearranges terms to obtain, $x^2 = 16$ $\Rightarrow x = +4$ or -4	**1.5**
	Ignores *x* = -4 as the length of PQ cannot be negative and writes ∠RPS will be maximum when length of PQ = 4 cm.	**0.5**
11	**Writes the marginal cost function as:** $M(n) = \frac{dC}{dn} = \frac{d}{dn}(400 + 4n + 0.0001n^2)$	**0.5**
	Finds the marginal cost function by completing the differentiation in the above step as: $M(n) = 4 + 0.0002n$	**1**
	Finds the marginal cost of 10 earphone sets as 4.002 dollars by substituting 10 for *n* in the equation obtained in the above step.	**0.5**
12	**Writes the daily profit function as:** $P(n) = (10 - 0.0004n)n - (400 + 4n + 0.0001n^2)$	**0.5**

Q.No	What to look for	Marks
	Simplifies the above equation to find the daily profit function as: $P(n) = -0.0005n^2 + 6n - 400$	**0.5**
	Differentiates the daily profit function obtained in the previous step as follows: $P'(n) = \frac{d}{dn}(-0.0005n^2 + 6n - 400)$ $= -0.001n + 6$	**0.5**
	Finds the critical point of $P(n)$ as $n = 6000$ by equating $P'(n)$ to 0 and solving for n as shown below: $P'(n) = -0.001n + 6 = 0$ $\Rightarrow n = 6000$	**0.5**
	Finds the second derivative as: $P''(n) = -0.001$ **Writes that this means that the function will be at its maximum at $n = 6000$.**	**0.5**
	Concludes that since the maximum value is obtained at $n = 6000$, hence the profit is maximized when the daily production is 6000 earphones.	**0.5**